Delay-dependent sampled-data control based on delay estimates

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SAR - 26/11 2010

Outline

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Motivating problem : Digital control

Classical control loop

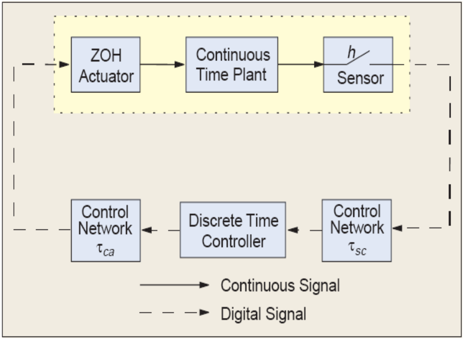


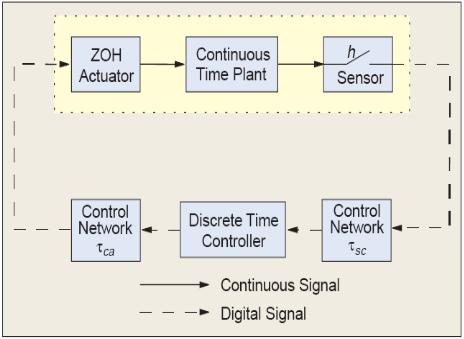
fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

Ideal Hypothesis :

- Sampling and actuation are periodic and synchronous

Motivating problem : Digital control over networks

Classical control loop



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Real-time problem : the system is affected by timing problems

- sampling jitter (sensor, multitasking processors)
- delays jitter (e.g. network delay)

(Wittenmark, Nilsson, Torngren, 1995)

Existing work

Continuous-time :

- Hespanha, Naghshtabrizi, Proceeding IEEE, 2007

Discrete-time :

- Zhang, Branicky, Phillips, IEEE Contr. Syst. Mag. 2001

Delay compensation method (Zhang et al., 2001)

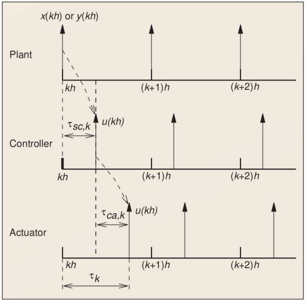


fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

Continuous-time model

$$\dot{x}(t) = Ax(t) + Bu(t), \ \tau_k \in [0, T]$$

Delay effect on control

$$u(t) = \begin{cases} u_{k-1}, & \forall t \in [kT, kT + \tau_k) \\ u_k, & \forall t \in [kT + \tau_k, (k+1)T) \,. \end{cases}$$

Delay compensation method (Zhang et al., 2001)

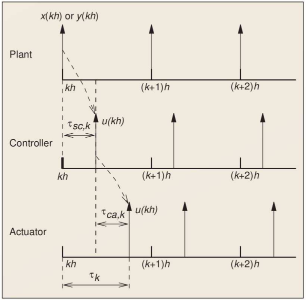


fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

Discrete-time model with delay :

$$x_{k+1} = A_d x_k + \Omega(T - \tau_k) B u_k + (B_d - \Omega(T - \tau_k) B)\, u_{k-1}$$

$$A_d = e^{AT},\ \ B_d = \int_0^T e^{As} ds B,\ \ \Omega(\tau) := \int_0^\tau e^{As} ds.$$

Delay compensation method (Zhang et al., 2001)

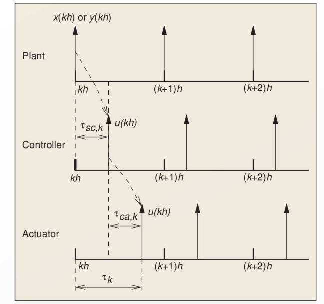


fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

- **Assumption :** the value of the delay τ_k is known
- **Idea :** use the information for estimating $x(kT + \tau_k)$

$$\bar{x}_k = x\left(kT + \tau_k\right) = e^{A\tau_k} x_k + \int_0^{\tau_k} e^{As} ds B u_{k-1}$$

Delay compensation method (Zhang et al., 2001)

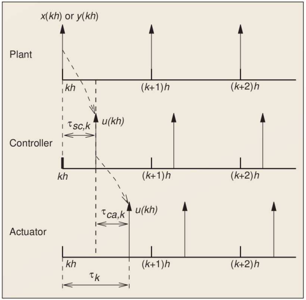


fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

- Delay-dependent control law

$$u_k = K\bar{x}_k = Kx\left(kT + \tau_k\right) = Ke^{A\tau_k}x_k + K\int_0^{\tau_k} e^{As}ds B u_{k-1}$$

Delay compensation method (Zhang et al., 2001)

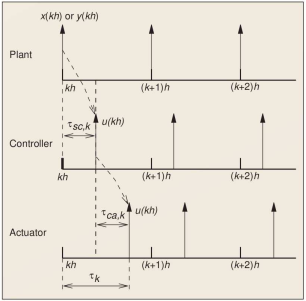


fig. from Zhang, et al. , IEEE Contr. Syst. Mag. 2001

- Closed-loop system at actuation times (delay free system)

$$\bar{x}_{k+1} = \left(e^{A(T+\tau_{k+1}-\tau_k)} + \int_0^{(T+\tau_{k+1}-\tau_k)} e^{As} ds BK \right) \bar{x}_k$$

(easy to check stability when the delay is constant)

Delay compensation method : practical problems

- the delay is not constant $\Rightarrow$ difficulty in the choice of K

$$\begin{aligned} \bar{x}_{k+1} &= \left(e^{A(T+\tau_{k+1}-\tau_k)} + \int_0^{(T+\tau_{k+1}-\tau_k)} e^{As} ds BK \right) \bar{x}_k \\ &= \tilde{\Phi}(T+\tau_{k+1}-\tau_k)\bar{x}_k \quad \text{(LTV model)} \end{aligned}$$

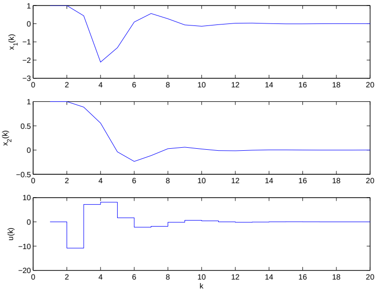
- the delay values is not exactly known

$$\hat{\tau}_k = \tau_k + \delta\tau_k$$

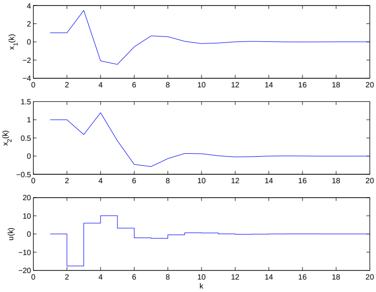
with bounded error $\delta\tau_k$

Delay compensation method : practical problems

1) Delay variation



$\tau = 0.006s$



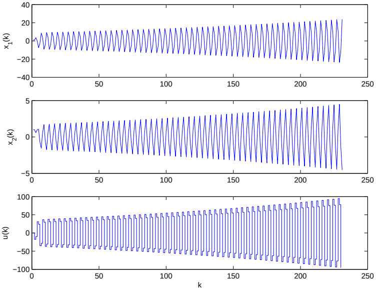
$\tau = 0.012s$

$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad K = -[4.57 \quad 3.02]$$

$$T = 0.012$$

Delay compensation method : practical problems

1) Delay variation



$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad K = -[4.57 \quad 3.02]$$

$$T = 0.012s$$

Delay compensation method : practical problems

2) Uncertain delay knowledge :

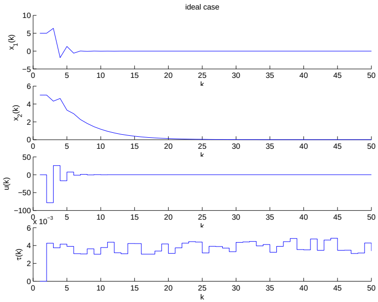


FIG.: Ideal evolution : $\delta\tau_k = 0$

$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad K = -[10 \;\; 0.13]$$

$$T = 0.005s$$

Delay compensation method : practical problems

2) Uncertain delay knowledge :

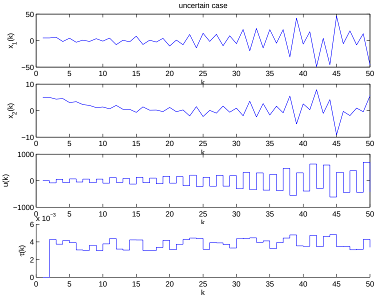


FIG.: Evolution with uncertainty : $\delta\tau_k = T/10$

$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad K = -[10 \quad 0.13]$$

$$T = 0.005s$$

Goal

Provide a robust delay compensation method !

Goal

- Control law in (Zhang et al., 2001)

$$\begin{aligned} u(t) &= K e^{A\tau_k} x_k + K \int_0^{\tau_k} e^{As} ds B u_{k-1}, \\ &= K_x(\tau_k) x_k + K_u(\tau_k) u_k \end{aligned}$$

$$\forall t \in [kT + \tau_k, (k+1)T + \tau_{k+1})$$

(complex structure $\Rightarrow$ difficult design problem)

- Proposed delay-dependent control law

$$u_{k+1} = K_x(\hat{\tau}_k) x_k + K_u^0(\hat{\tau}_k) u_k + K_u^1(\hat{\tau}_k) u_{k-1}$$

Goal : provide LMI for robust design !

Open-loop model

- Discrete-time model (integration over a sampling period)

$$x_{k+1} = A_d x_k + \Omega(T - \tau_k) B u_k + \left(B_d - \Omega(T - \tau_k) B\right) u_{k-1}$$

$$A_d = e^{AT}, \; B_d = \int_0^T e^{As} ds B, \; \Omega(\tau) := \int_0^\tau e^{As} ds.$$

- Control law

$$u_{k+1} = K_x(\hat{\tau}_k) x_k + K_u^0(\hat{\tau}_k) u_k + K_u^1(\hat{\tau}_k) u_{k-1}$$

with

$$\hat{\tau}_k = \tau_k + \delta\tau_k, \; \delta\tau_{min} \leq \delta\tau_k \leq \delta\tau_{max}$$

Augmented state model

Consider $\eta_k = \left[x_k' \;\, u_{k-1}' \;\, u_k'\right]'$.

$$\eta_{k+1} = \bar{A}(\tau_k)\eta_k + \bar{B}v_k$$

Delay-free LPV model

$$\bar{A}(\tau_k) = \begin{bmatrix} A_d & B_d - \Omega(T-\tau_k)B & \Omega(T-\tau_k)B \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

with

$$v_k = \mathcal{K}(\hat{\tau}_k)\eta_k$$

and

$$\mathcal{K}(\hat{\tau}_k) = \left[K_x(\hat{\tau}_k) \;\, K_u^1(\hat{\tau}_k) \;\, K_u^0(\hat{\tau}_k)\right]$$

Closed-loop model

$$\eta_{k+1} = \left(\bar{A}(\tau_k) + \bar{B}\mathcal{K}(\hat{\tau}_k)\right)\eta_k$$

(depends both on τ_k, and $\hat{\tau}_k = \tau_k + \delta\tau_k$)

$$\eta_{k+1} = \left(\bar{A}(\hat{\tau}_k) + \bar{B}\mathcal{K}(\hat{\tau}_k)\right)\eta_k + E\Omega\left(\delta\tau_k\right)\mathcal{F}(\hat{\tau}_k)\eta_k$$

where

$$E\Omega\left(\delta\tau_k\right)\mathcal{F}(\hat{\tau}_k) = \bar{A}(\tau_k) - \bar{A}(\hat{\tau}_k)$$

and

$$\Omega(\delta\tau) := \int_0^{\delta\tau} e^{As} ds.$$

Property : $\exists\ \gamma > 0\ s.t.\ \| \Omega(\delta\tau_k) \| \leq \gamma^2,\ \forall\ \delta\tau_k \in [\delta\tau_{min}, \delta\tau_{max}]$

Parametric set of LMI

Theorem

The system is stabilizable if there exist a positive scalar λ*, symmetric positive definite matrices* $\mathcal{S}(\cdot)$ *and matrices* $\mathcal{R}(\cdot), \mathcal{G}(\cdot)$*, s.t. the following set of LMI is feasible :*

$$\begin{bmatrix} \mathcal{G}(\theta) + \mathcal{G}'(\theta) - \mathcal{S}(\theta) & \mathcal{G}'(\theta)\bar{A}'(\theta) + \mathcal{R}'(\theta)\bar{B}' & \mathcal{G}'(\theta)\mathcal{F}'(\theta) \\ * & \mathcal{S}(\theta_+) - \lambda E E' \gamma^2 & \mathbf{0} \\ * & * & \lambda \mathbf{I} \end{bmatrix} > 0$$

for all scalars $(\theta, \theta_+) \in [0, T]^2$*. The control law is given with* $\mathcal{K}(\hat{\tau}_k) = \mathcal{R}(\hat{\tau}_k) \left(\mathcal{G}(\hat{\tau}_k)\right)^{-1}$.

$$V(\eta_k, \hat{\tau}_k) = \eta_k' \left(\mathcal{S}\left(\hat{\tau}_k\right)\right)^{-1} \eta_k$$

(Lyapunov function)

Quantization Approach

- Consider a gridding of the domain $[0, T]$, in N subdomains bounded by the values $\theta_i = i \times \dfrac{T}{N}$.
- **Assumption :** τ_k is uncertain, but we know the range in which we take values $[\theta_i, \theta_{i+1}) \subseteq [0, T]$.
- Use a finite set of controller gains $\mathcal{K}_i$ for $\tau_k \in [\theta_i, \theta_{i+1}) \subseteq [0, T]$
- $\delta\tau_k \in \left[-\dfrac{T}{2N}, \dfrac{T}{2N}\right] \Rightarrow \gamma$ s.t. $\parallel \Omega(\delta\tau_k) \parallel \leq \gamma^2$

Quantization Approach

- Gridding of the domain $[0, T]$, in N subdomains bounded by the values $\theta_i = i \times \dfrac{T}{N}$.

- Finite set of LMI conditions

$$\begin{bmatrix} \mathcal{G}_i + \mathcal{G}_i' - \mathcal{S}_i & \mathcal{G}_i'\bar{A}'\left(\dfrac{\theta_{i+1}+\theta_i}{2}\right) + \mathcal{R}_i'\bar{B}' & \mathcal{G}_i'\mathcal{F}'\left(\dfrac{\theta_{i+1}+\theta_i}{2}\right) \\ * & \mathcal{S}_j - \lambda E E'\gamma^2 & \mathbf{0} \\ * & * & \lambda \mathbf{I} \end{bmatrix} > 0,$$

$\forall (i,j) \in \{0, \ldots, N-1\}^2$.

$$\mathcal{K}_i = \mathcal{R}_i \left(\mathcal{G}_i\right)^{-1},$$

Example of control design for the Quantization Approach

- System
$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
with $a = 1$ and $b = -15$
- unstable open-loop matrix A, complex eigenvalues $\lambda = 1 \pm 15i$.
- sampling period $T = 0.09s$.
- No stabilizing state feedback possible (Hetel, 2006, IEEE Trans. Autom. Contr. ; Cloosterman, Automatica, 2010)
- $\gamma = 1 - e^{-\delta \tau_{min}}$

Example of control design for the Quantization Approach

- System
$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
with $a = 1$ and $b = -15$
- $[0, T]$ is divided in 3 subintervals
- $\hat{\tau}_k \in \{0.015, 0.045, 0.075\}$, $\delta\tau_{min} = -0.015$, $\delta\tau_{max} = 0.015$
- Gains

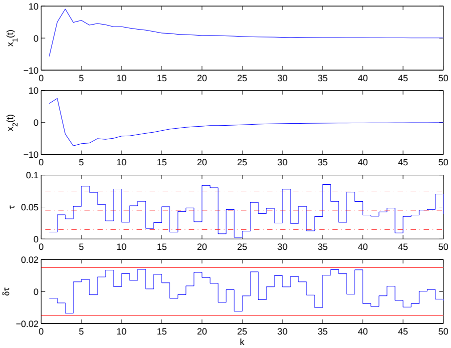
$$\begin{aligned} \mathcal{K}_1 &= [9.90\ 9.03\ 0.27\ 0.68], \\ \mathcal{K}_2 &= [9.93\ 9.03\ 0.76\ 0.19], \\ \mathcal{K}_3 &= [9.95\ 9\ 1.07\ -0.12] \end{aligned}$$

Example of control design for the Quantization Approach

- System

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $a = 1$ and $b = -15$



Polytopic Approach

- LMI condition

$$\begin{bmatrix} \mathcal{G}(\theta) + \mathcal{G}'(\theta) - \mathcal{S}(\theta) & \mathcal{G}'(\theta)\bar{A}'(\theta) + \mathcal{R}'(\theta)\bar{B}' & \mathcal{G}'(\theta)\mathcal{F}'(\theta) \\ * & \mathcal{S}(\theta_+) - \lambda E E' \gamma^2 & \mathbf{0} \\ * & * & \lambda \mathbf{I} \end{bmatrix} > 0$$

$(\theta, \theta_+) \in [0, T]^2$

- Matrix with exponential uncertainty $\Omega(T-\theta) = \int_0^{T-\theta} e^{As} ds$

$$\bar{A}(\theta) = \begin{bmatrix} A_d & B_d - \Omega(T-\theta)B & \Omega(T-\theta)B \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

Polytopic Approach

- Matrix with exponential uncertainty
  $\Omega(T-\theta) = \int_0^{T-\theta} e^{As} ds, \theta \in [0, T]$
- Polytopic embedding of exponential uncertainty

$$\Omega(T-\theta) \in \mathcal{W} = co\left\{\Omega_1, \Omega_2, \ldots, \Omega_p\right\}.$$

- for all $\theta \in \mathcal{T}$ the matrix $\Omega(T-\theta)$ can be expressed as

$$\Omega(T-\theta) = \sum_{i=1}^{p} \mu_i(\theta)\Omega_i, \ \sum_{i=1}^{p} \mu_i(\theta) = 1, \mu_i(\theta) > 0, \ \forall i = 1, \ldots, p.$$

- the parameters $\mu_i(\hat{\tau}), \ i = 1, \ldots, p$ represent the barycentric coordinates of the matrix $\Omega(T-\hat{\tau})$ in the polytope $\mathcal{W}$.

(Hetel, Daafouz, Iung, IEEE Trans. Autom. Contr. 2006)

Polytopic Approach

- Matrix with exponential uncertainty

$$\Omega(T-\theta) = \int_0^{T-\theta} e^{As} ds, \theta \in [0, T]$$

$$\Omega(T-\theta) = \sum_{i=1}^{p} \mu_i(\theta)\Omega_i, \ \sum_{i=1}^{p} \mu_i(\theta) = 1, \mu_i(\theta) > 0, \ \forall i = 1, \dots, p.$$

- By convexity, it suffice to test the LMI conditions on the vertex $\Omega_i \Rightarrow$ finite number of conditions
- Polytopic feedback

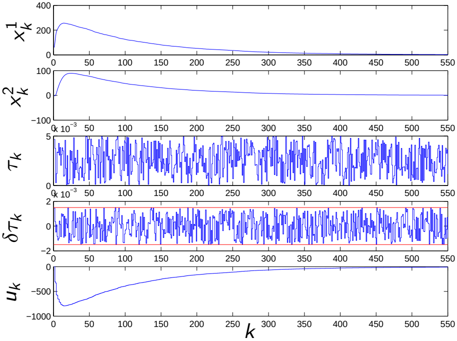
$$u_{k+1} = \sum_{i=1}^{p} \mu_i(\hat{\tau}_k) \left(K_x^i x_k + K_u^{0^i} u_k + K_u^{1^i} u_{k-1} \right)$$

Example of control design for the Polytopic Approach

Motivating Example (unstable under uncertainty and delay variation)

$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad T = 0.005s,$$

$$\delta\tau_k \in [-0.0015,\ 0.0015].$$



Polytopic feedback with 50 vertex.

Conclusion

- LTI systems with feedback delay
- Numerical methods for the design of a delay-dependent sampled-data state feedback.
- The controller is adapted in real time according to an estimate of the delay value.
- Robustness with respect to the delay uncertainty

References

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Stability for the Compensation Method in Zhang, 2001

- Discrete-time model (integration over a sampling period)

$$x_{k+1} = A_d x_k + \Omega(T-\tau_k)Bu_k + \left(B_d - \Omega(T-\tau_k)B\right)u_{k-1}$$

$$A_d = e^{AT},\ B_d = \int_0^T e^{As}dsB,\ \Omega(\tau) := \int_0^\tau e^{As}ds.$$

- Control law

$$\begin{aligned} u_k &= Ke^{A\tau_k}x_k + K\int_0^{\tau_k} e^{As}dsBu_{k-1}, \\ &= K_x(\tau_k)x_k + K_u(\tau_k)u_k \end{aligned}$$

with

$$\hat{\tau}_k = \tau_k + \delta\tau_k,\ \delta\tau_{min} \leq \delta\tau_k \leq \delta\tau_{max}$$

Stability for the Compensation Method in Zhang, 2001

Augmented Model

$$\zeta_{k+1} = \tilde{A}(\tau_k, \hat{\tau}_k)\zeta_k,$$

$$\tilde{A}(\tau_k, \hat{\tau}_k) = \begin{bmatrix} A_d + \Omega(T - \tau_k)BK_x(\hat{\tau}_k) & B_d + \Omega(T - \tau_k)B\left(K_u(\hat{\tau}_k) - \mathbf{I}\right) \\ K_x(\hat{\tau}_k) & K_u(\hat{\tau}_k) \end{bmatrix},$$

$$\zeta_k = \left[x_k' \;\; u_{k-1}'\right]', \;\; K_x(\hat{\tau}_k) = Ke^{A\hat{\tau}_k}, \;\; K_u(\hat{\tau}_k) = K\int_0^{\hat{\tau}_k} e^{As}dsB.$$

$$\tilde{A}'(\tau_k, \hat{\tau}_k)P\tilde{A}(\tau_k, \hat{\tau}_k) - P \prec 0, \;\; P \succ 0$$